BOARD OF STUDIES
NEW SOUTH WALES

2012 HSC Mathematics
'Sample Answers'

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Section II

Question 11 (a)

$2x^2 - 7x + 3 = (2x - 1)(x - 3)$

Question 11 (b)

$|3x - 1| < 2$

$-2 < 3x - 1 < 2$

$-1 < 3x < 3$

Hence $-\frac{1}{3} < x < 1$

Question 11 (c)

$y = x^2, \qquad \frac{dy}{dx} = 2x$

slope of tangent at $x = 3$ is $2 \times 3 = 6$

$\therefore 6 = \frac{y - 3^2}{x - 3} = \frac{y - 9}{x - 3}$

Hence the equation of the tangent is $y = 6(x - 3) + 9$

$= 6x - 9$

Question 11 (d)

$y' = 5\left(3 + e^{2x}\right)^4 \times 2e^{2x}$

$= 10e^{2x}\left(3 + e^{2x}\right)^4$

Question 11 (e)

$x^2 = 16(y-2) = 4.4\,(y-2)$, so $a = 4$

the vertex is at $(0, 2)$, so the focus is at $(0, 2+4) = (0, 6)$

Question 11 (f)

Area of the sector is given by $A = \frac{\theta}{2} r^2$

$$\begin{aligned} \text{ie } 50 &= \frac{\theta}{2} r^2 \\ &= \frac{\theta}{2} \cdot 6^2 \\ &= \frac{\theta}{2} \cdot 36 \\ &= 18\theta \\ \therefore\ \theta &= \frac{50}{18} \\ \text{now, } l &= r\theta \\ &= 6 \times \frac{50}{18} \\ \text{length of arc} &= \frac{50}{3} \text{ cm} \end{aligned}$$

Question 11 (g)

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\, dx &= \left[2\tan\frac{x}{2} \right]_0^{\frac{\pi}{2}} \\ &= 2\tan\frac{\pi}{4} - 2\tan 0 \\ &= 2 \times 1 - 0 \\ &= 2 \end{aligned}$$

Question 12 (a) (i)

$$y' = \log_e x + (x-1)\frac{1}{x}$$
$$= \log_e x + 1 - \frac{1}{x}$$

Question 12 (a) (ii)

$$y' = \frac{-x^2 \sin x - 2x\cos x}{x^4}$$
$$= \frac{-(x\sin x + 2\cos x)}{x^3}$$

Question 12 (b)

$$\int \frac{4x}{x^2+6}\,dx = 2\int \frac{2x}{x^2+6}\,dx$$
$$= 2\log_e(x^2+6) + C$$

Question 12 (c) (i)

Every row has two tiles more than the previous row and the first row has three tiles.

It is an arithmetic sequence and $T_{20} = 3 + 19 \times 2$

$= 41$

ie There are 41 tiles in row 20.

Question 12 (c) (ii)

The number of tiles for the 20 rows is

$$\begin{aligned} S_{20} &= \frac{20}{2}\left(3 + T_{20}\right) \\ &= 10(3 + 41) \\ &= 440 \end{aligned}$$

Question 12 (c) (iii)

We want $\frac{n}{2}\left(3 + T_n\right) = 200$,

where $T_n = 3 + 2(n - 1) = 2n + 1$.

Hence

$$\frac{n}{2}(3 + 2n + 1) = 200$$

$$\text{ie} \quad n(n + 2) = 200$$

$$n^2 + 2n - 200 = 0$$

$$\begin{aligned} \therefore n &= \frac{-2 \pm \sqrt{4 + 800}}{2} \\ &= -1 \pm \sqrt{201} \\ &= 13.1774 \end{aligned}$$

Hence Jay can make 13 complete rows.

Question 12 (d) (i)

If $f(x)$ denotes depth at distance x from the river bank then, by Simpson's rule, the approximate area is:

$$\begin{aligned} A &= \frac{3}{3}\left(f(0) + 4f(3) + 2f(6) + 4f(9) + f(12)\right) \\ &= 1(0.5 + 4 \times 2.3 + 2 \times 2.9 + 4 \times 3.8 + 2.1) \end{aligned}$$

$\therefore$ area $= 32.8\ \text{m}^2$

Question 12 (d) (ii)

Volume through the cross-section in 10 seconds is

$(32.8 \times 0.4 \times 10)\text{m}^3 = 131.2 \text{ m}^3$

Question 13 (a) (i)

Coordinates of A : $y = 0, \quad 2x = 8, \quad$ so
$x = 4$
ie $A(4, 0)$
coordinates of B : $x = 0, \quad y = 8$
ie $B(0, 8)$

By Pythagoras' theorem
$$\begin{aligned} AB &= \sqrt{4^2 + 8^2} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \end{aligned}$$

Question 13 (a) (ii)

By the cosine rule

$$AC^2 = AB^2 + BC^2 - 2AB \times BC \cos(\angle ABC)$$
$$25 = 80 + 65 - 2 \times 4\sqrt{5} \times \sqrt{65} \cos(\angle ABC)$$
$$40\sqrt{13} \cos(\angle ABC) = 120$$
$$\cos(\angle ABC) = \frac{3}{\sqrt{13}}$$

Hence $\quad \angle ABC \approx 33.69^\circ$,

so the angle is 34° to the nearest degree.

Question 13 (a) (iii)

Slope of AB is -2, so the slope of CN is $\frac{1}{2}$.

Equation of CN is $\frac{1}{2} = \frac{y-4}{x-7}$,

$$\text{so } y = \frac{1}{2}(x-7) + 4$$

$$= \frac{x}{2} + \frac{1}{2}.$$

The coordinates of N are obtained by the intersection of AB and CN:

$$y = -2x + 8 = \frac{1}{2}x + \frac{1}{2}$$

$$-4x + 16 = x + 1$$

$$15 = 5x$$

$$3 = x$$

Since $y = -2x + 8$

$$y = -6 + 8$$

$$= 2$$

Hence N has coordinates $(3, 2)$

Question 13 (b) (i)

For the x-coordinate at the intersection of the two parabolas

$$x^2 - 3x = 5x - x^2$$

$$2x^2 - 8x = 0$$

$$2x(x-4) = 0$$

$$x = 0, 4$$

so $x = 4$ is the x-coordinate of point A.

Question 13 (b) (ii)

Area is given by

$$\int_0^4 \left(5x - x^2\right) - \left(x^2 - 3x\right) dx$$

$$= \int_0^4 8x - 2x^2 \, dx$$

$$= \left[4x^2 - \frac{2}{3}x^3\right]_0^4$$

$$= 64 - \frac{128}{3}$$

$$= \frac{192 - 128}{3}$$

$$\text{area } = \frac{64}{3} \text{ units}^2$$

Question 13 (c) (i)

$$\frac{3}{5} \times \frac{3}{7} = \frac{9}{35}$$

Question 13 (c) (ii)

Complement of (i) : $1 - \frac{9}{35} = \frac{26}{35}$

Question 13 (c) (iii)

Probability of 2 red + probability of 2 white:

$$\frac{9}{35} + \frac{2}{5} \times \frac{4}{7} = \frac{17}{35}$$

Question 14 (a) (i)

$f'(x) = 12x^3 + 12x^2 - 24x$

$\quad = 12x(x^2 + x - 2)$

$\quad = 12x(x+2)(x-1)$

Hence the stationary points are at $x = 0, x = 1, x = -2$

Now $f(0) = 0,\ f(1) = 3 + 4 - 12$ and $f(-2) = 316 - 48 - 124$

$\quad = -5 \qquad\qquad = -32$

Hence the statinary points are

$(-2, -32), (0, 0), (1, -5)$

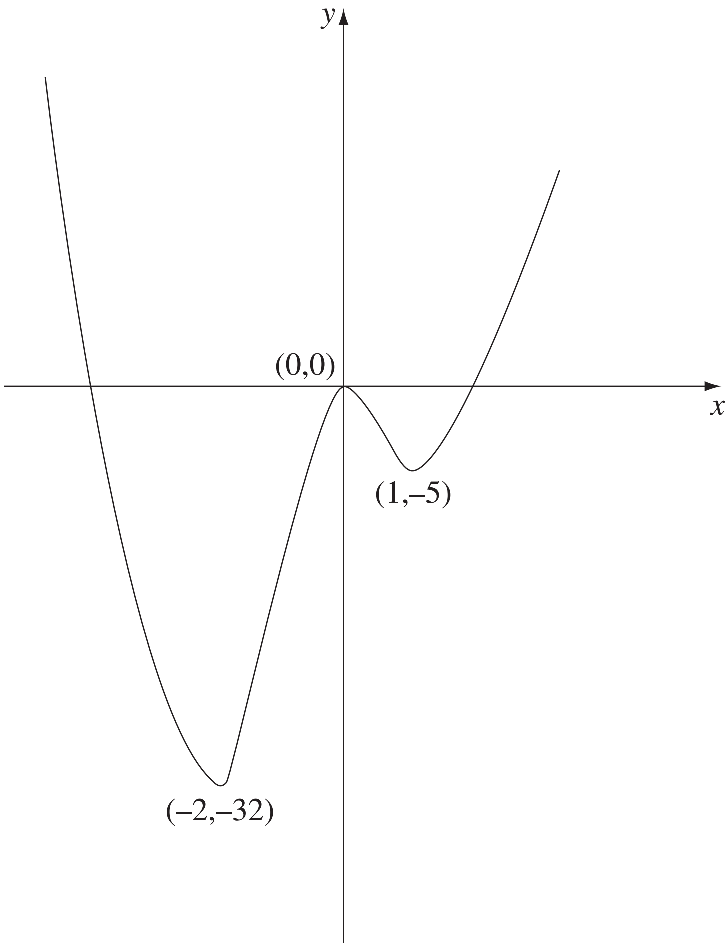
Now $f''(x) = 12(3x^2 + 2x - 2)$

$f''(0) = -24 < 0,$ so $(0, 0)$ is a maximum

$f''(1) = 12 \times 3 > 0,$ so $(1, -5)$ is a minimum

$f''(-2) = 12 \times 6 > 0,$ so $(-2, -32)$ is a minimum

Question 14 (a) (ii)



Question 14 (a) (iii)

$f(x)$ is increasing for $-2 < x < 0$ or for $x > 1$

Question 14 (a) (iv)

k is the vertical shift of the graph of f.
To make sure the equation has no solution (ie the new graph should not cut the x-axis) move the graph up by the smallest minimum, so $k > 32$.

Question 14 (b)

Volume is given by $V = \int \pi y^2 \, dx$

$$
\begin{aligned}
V &= \pi \int_0^1 \frac{9}{(x+2)^4} \, dx \\
&= 9\pi \int_0^1 (x+2)^{-4} \, dx \\
&= \left[-\frac{9\pi}{3} (x+2)^{-3} \right]_0^1 \\
&= -3\pi \left(3^{-3} - 2^{-3} \right) \\
&= -3\pi \left(\frac{1}{27} - \frac{1}{8} \right)
\end{aligned}
$$

$\therefore$ Volume $= \dfrac{19\pi}{72}$ units^3

Question 14 (c) (i)

$$
\begin{aligned}
N(20) = 1000e^{20k} &= 2000 \\
e^{20k} &= 2 \\
20k &= \ln 2 \\
k &= \frac{\ln 2}{20} \approx 0.0347
\end{aligned}
$$

Question 14 (c) (ii)

$$
\begin{aligned}
N(120) &= 1000e^{120k} \\
&= 1000\, e^{120 \times 0.0347}
\end{aligned}
$$

Number of bacteria $\approx 64\,328$

Question 14 (c) (iii)

$\frac{dN}{dt} = kN$, so from (ii)

$\frac{dN}{dt} = 0.0347 \times 64\,328 \approx 2232$ when $t = 120$

rate of change ≈ 2232 bacteria/minute

Question 14 (c) (iv)

At $t = 0 \quad N = 1000$

Find t so that $100\,000 = 1000e^{kt}$

$$100 = e^{kt}$$

$$\ln 100 = kt$$

Hence $$t = \frac{\ln 100}{k}$$

$$= \frac{\ln 100}{0.0347}$$

time ≈ 132.7 minutes

Question 15 (a) (i)

Length in cm is

$10 + 10 \times 0.96 + 10 \times 0.96^2 + \ldots + 10 \times 0.96^9$

$= 10\left(1 + 0.96 + \ldots + 10 \times 0.96^9\right)$

$= 10\left(\dfrac{1 - 0.96^{10}}{1 - 0.96}\right)$

≈ 83.79

Question 15 (a) (ii)

Since $0.96 < 1$ the limiting sum $10\left(1 + 0.96 + 0.96^2 + \ldots\right)$ exists.

The limiting sum is $10\left(\dfrac{1}{1 - 0.96}\right) = \dfrac{10}{0.04} = 250$

As 250 cm $<$ 300 cm, a strip of length 3 m is sufficient.

Question 15 (b) (i)

Initial velocity is $\dot{x}(0) = 1 - 2\cos 0$

$= -1$ m/s

Question 15 (b) (ii)

$\ddot{x} = 2\sin t = 0$ if $t = 0,\ \pi,\ 2\pi,\ \ldots$

The first maximum velocity is at $t = \pi$

$\dot{x}(\pi) = 1 - 2\cos\pi$

$= 3$ m/s

Question 15 (b) (iii)

$$x = \int \dot{x}\ dt = \int 1 - 2\cos t\, dt$$

$$= t - 2\sin t + C$$

We are given that $x(0) = 3$, so

$x(0) = -2\sin 0 + C = 3$

Hence $C = 3$ and the displacement is

$$x = t - 2\sin t + 3$$

Question 15 (b) (iv)

The particle is at rest if $\dot{x} = 0$, so

$\dot{x} = 1 - 2\cos t = 0$

ie $\cos t = \frac{1}{2}$

Hence $t = \frac{\pi}{3}$

The displacement is $x\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3} - 2\sin\frac{\pi}{3} + 3\right)$ metres

$= \left(\frac{\pi}{3} - \sqrt{3} + 3\right)$ metres

Question 15 (c) (i)

$$A_2 = \left[360\,000(1 + 0.005) - M\right](1 + 0.005) - M$$

$$= 360\,000(1.005)^2 - M(1 + 1.005)$$

Question 15 (c) (ii)

Generalising from (i)

$$A_n = 360\,000(1.005)^n - M\left(1 + 1.005 + \ldots + 1.005^{n-1}\right)$$

$$= 360\,000(1.005)^n - M\frac{\left(1.005^n - 1\right)}{(1.005 - 1)}$$

We require $A_{300} = 0$, so

$$360\,000(1.005)^{300} = M\frac{\left(1.005^{300} - 1\right)}{(1.005 - 1)}$$

$$M = \frac{360\,000(1.005)^{300}\,0.005}{\left(1.005^{300} - 1\right)} \approx 2319.50$$

Question 15 (c) (iii)

We want to find the smallest n so that $A_n < 180\,000$

$$360\,000(1.005)^n - M\frac{\left(1.005^n - 1\right)}{0.005} = 180\,000$$

$$360\,000(1.005)^n - 463\,900\left(1.005^n - 1\right) = 180\,000$$

$$103\,900(1.005)^n = 283\,900$$

$$(1.005)^n = 2.7324$$

Hence $n = \dfrac{\log 2.7324}{\log 1.005} = 201.5$

After 202 months $\$A_n$ will be less than \$180 000 for the first time.

Question 16 (a) (i)

$EF \,||\, CD$ since $CDEF$ is a rhombus

$ED \,||\, FC$ since $CDEF$ is a rhombus

$\angle FEB = \angle DAE$ (corresponding angles, $EF \parallel CA$)

$\angle FBE = \angle DEA$ (corresponding angles, $ED \parallel BC$)

Hence $\triangle EBF$ is similar to $\triangle AED$ since two (and therefore all) angles are equal.

Question 16 (a) (ii)

Using that $\triangle EBF$ is similar to $\triangle AED$,

$\frac{x}{a-x} = \frac{b-x}{x}$ (corresponding sides of similar triangles)

$x^2 = (b-x)(a-x)$

$x^2 = ba - ax - bx + x^2$

$0 = ba - x(a+b)$

$x = \frac{ab}{a+b}$

Question 16 (b) (i)

T has coorindates $(\cos\theta, \sin\theta)$

The line OT has slope $\frac{\sin\theta}{\cos\theta}$

Hence the line PT perpendicular to OT has slope $-\frac{\cos\theta}{\sin\theta}$ and passes through T.

Hence the equation of PT is:

$$-\frac{\cos\theta}{\sin\theta} = \frac{y-\sin\theta}{x-\cos\theta}$$

$$-x\cos\theta + \cos^2\theta = y\sin\theta - \sin^2\theta$$

$$\begin{aligned} x\cos\theta + y\sin\theta &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

Question 16 (b) (ii)

Q is the point of intersection of the line $y = 1$ with the line from (i).

Hence the x-coordinates of Q satisfies

$$x\cos\theta + 1\sin\theta = 1$$

$$x = \frac{1-\sin\theta}{\cos\theta}$$

The length of BQ is $\dfrac{1-\sin\theta}{\cos\theta}$

Question 16 (b) (iii)

Area of trapezium is given by

$$A = \frac{1}{2}OB\left(OP + BQ\right)$$

P is on the line $x\cos\theta + y\sin\theta = 1$ with $y = 0$,

so $x = \dfrac{1}{\cos\theta}$ ie $OP = \dfrac{1}{\cos\theta}$

$OB = 1$ and from (ii) $BQ = \dfrac{1-\sin\theta}{\cos\theta}$

$$\therefore\ A = \frac{1}{2}\left(\frac{1}{\cos\theta} + \frac{1-\sin\theta}{\cos\theta}\right)$$

$$= \frac{1}{2}\left(\frac{2-\sin\theta}{\cos\theta}\right)$$

$$= \frac{2-\sin\theta}{2\cos\theta}$$

Question 16 (b) (iv)

Differentiate area with respect to θ:

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\left(\frac{2-\sin\theta}{2\cos\theta}\right)$$

$$= \frac{-\cos^2\theta + (2-\sin\theta)\sin\theta}{2\cos^2\theta}$$

$$= \frac{2\sin\theta - \left(\cos^2\theta + \sin^2\theta\right)}{2\cos^2\theta}$$

$$= \frac{2\sin\theta - 1}{2\cos^2\theta}$$

Need to solve $2\sin\theta - 1 = 0$

ie $\sin\theta = \frac{1}{2}$

Hence $\theta = \frac{\pi}{6}$ is a critical point

If $\theta \to \frac{\pi}{2}$ then the area of the trapezium becomes very large: $A \to \infty$

If $\theta = 0$, then $\frac{dA}{d\theta} = -\frac{1}{2} < 0$, so the area is decreasing.

As there is only one stationary point it must be minimum.

Hence $\theta = \frac{\pi}{6}$ gives the minimum area.

Question 16 (c) (i)

Find the points of intersection of the parabola $y = x^2$ and a circle $x^2 + (y-c)^2 = r^2$:

$$y + (y-c)^2 = r^2$$
$$y + y^2 - 2cy + c^2 = r^2$$
$$y^2 + (1-2c)y + c^2 - r^2 = 0$$

The circle is tangent if there is precisely one solution, so the discriminant has to vanish.

$$(1-2c)^2 - 4(c^2 - r^2) = 0$$
$$(1-2c)^2 = 4(c^2 - r^2)$$
$$1 - 4c + 4c^2 = 4c^2 - 4r^2$$
$$4c = 1 + 4r^2 \text{ as required}$$

Question 16 (c) (ii)

y must be positive to be a solution since the circle is inside the parabola.

As the discriminant is zero

$$y = -\frac{1}{2}(1-2c) \geq 0$$
$$\text{so } 1 - 2c \leq 0$$
$$\frac{1}{2} \leq c$$

If $c = \frac{1}{2}$ there is only one point, so $c > \frac{1}{2}$.