BOARD OF STUDIES
NEW SOUTH WALES

2011 HSC Mathematics
'Sample Answers'

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This material is also provided to the Supervisor of Marking, to give some guidance about the nature and scope of the responses the committee expected students would produce. How sample answers are used at marking centres varies. Sample answers may be used extensively and even modified at the marking centre OR they may be considered only briefly at the beginning of marking. In a few cases, the sample answers may not be used at all at marking.

The Board publishes this information to assist in understanding how the marking guidelines were implemented.

The 'sample answers' or similar advice contained in this document are not intended to be exemplary or even complete answers or responses. As they are part of the examination committee's 'working document', they may contain typographical errors, omissions, or only some of the possible correct answers.

Question 1 (a)

$$\sqrt[3]{\frac{651}{4\pi}} = 3.727\ 838\ \ldots$$
$$= 3.728 \text{ (to 4 significant figures)}$$

Question 1 (b)

$$\frac{n^2 - 25}{n - 5} = \frac{\cancel{(n-5)}(n+5)}{\cancel{n-5}}$$
$$= n + 5$$

Question 1 (c)

$$2^{2x+1} = 32$$
$$2^{2x+1} = 2^5$$
$$2x + 1 = 5$$
$$x = 2$$

Question 1 (d)

$$\frac{d}{dx}\left(\ln\left(5x+2\right)\right)=\frac{5}{5x+2}$$

Question 1 (e)

$$\begin{aligned} 2-3x &\le 8 \\ -6 &\le 3x \\ x &\ge -2 \end{aligned}$$

Question 1 (f)

$$\begin{aligned} \frac{4}{\sqrt{5}-\sqrt{3}} &= \frac{4}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} \\ &= \frac{4\left(\sqrt{5}+\sqrt{3}\right)}{2} \\ &= 2\left(\sqrt{5}+\sqrt{3}\right) \end{aligned}$$

Question 1 (g)

$$\begin{aligned} \text{Exp. No.} &= 0.02\times 800 \\ &= 16 \end{aligned}$$

Question 2 (a) (i)

$$x^2 - 6x + 2 = 0$$

$$\begin{aligned} \alpha + \beta &= -\frac{b}{a} \\ &= \frac{6}{1} \\ &= 6 \end{aligned}$$

Question 2 (a) (ii)

$$\begin{aligned} \alpha\beta &= \frac{c}{a} \\ &= 2 \end{aligned}$$

Question 2 (a) (iii)

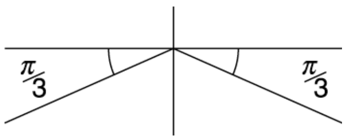
$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta + \alpha}{\alpha\beta} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

Question 2 (b)

$$2\sin x = -\sqrt{3} \qquad 0 \le x \le 2\pi$$

$$\sin x = -\frac{\sqrt{3}}{2}$$

$$x = \frac{4\pi}{3},\ \frac{5\pi}{3}$$



Question 2 (c)

$$y = (2x+1)^4$$

$$y' = 4 \times (2x+1)^3 \times 2$$

$$= 8(2x+1)^3$$

When $x = -1$, $y' = 8(-2+1)^3$

$$= -8$$

$$\therefore \quad y - y_1 = m(x - x_1)$$

$$y - 1 = -8(x - (-1))$$

$$y - 1 = -8x - 8$$

$$8x + y + 7 = 0$$

Question 2 (d)

$$\frac{dy}{dx} = x^2 \cdot e^x + e^x \cdot 2x$$

$$= e^x\left(x^2 + 2x\right)$$

Question 2 (e)

$$\int \frac{1}{3x^2}\,dx = \int \frac{1}{3}x^{-2}\,dx$$

$$= \frac{1}{3}\ \frac{x^{-1}}{-1} + C$$

$$= \frac{-1}{3x} + C$$

Question 3 (a) (i)

$\$3M, \$3.5M, \$4, \ldots$

$a = 3, d = 0.5$ arithmetic progression

$$T_n = a + (n-1)d$$
$$T_{25} = 3 + (25-1)0.5$$
$$= 15$$

$\therefore$ Cost = \$15 million

Question 3 (a) (ii)

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$S_{110} = \frac{110}{2}\left(2(3) + (110-1)0.5\right)$$
$$= 3\,327.5$$

$\therefore$ Total = \$3 327.5 million

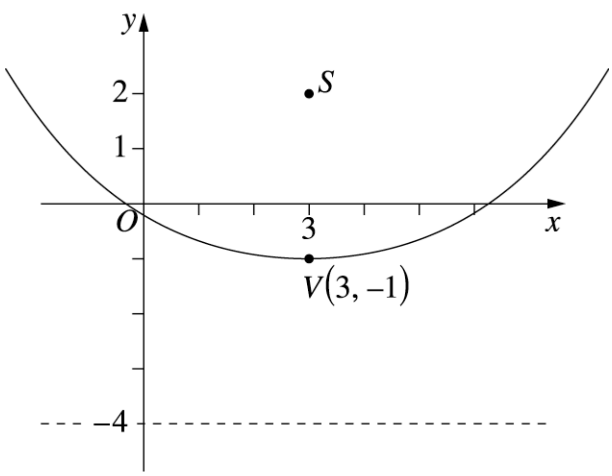
Question 3 (b)

$S(3,2)$ directrix $y = -4$

y-coordinate of the vertex is $\dfrac{2 + -4}{2} = -1$

x-coordinate is 3

Vertex is $(3,-1)$



Question 3 (c) (i)

$$3x + 4y - 12 = 0 \qquad \text{at } x = 0$$

$$3(0) + 4y - 12 = 0$$

$$4y = 12$$

$$y = 3 \qquad \therefore B(0,3)$$

Question 3 (c) (ii)

m_1 of $3x + 4y - 12 = 0$ is $-\dfrac{3}{4}$

m_2 of $4x - 3y = 0$ is $\dfrac{4}{3}$

$\therefore\ m_1 \times m_2 = -\dfrac{3}{4} \times \dfrac{4}{3} = -1$

$\therefore\ \ell_1 \perp \ell_2$

Question 3 (c) (iii)

$$OE = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| \qquad \text{(perpendicular distance formula)}$$

$$= \left| \frac{3(0) + 4(0) + -12}{\sqrt{3^2 + 4^2}} \right|$$

$$= \frac{12}{5}$$

Question 3 (c) (iv)

$$BE^2 = BO^2 - OE^2 \qquad \text{(Pythagoras' theorem since } \ell_1 \perp \ell_2\text{)}$$

$$= 3^2 - \left(\frac{12}{5}\right)^2$$

$$\therefore BE = \frac{9}{5}$$

Question 3 (c) (v)

$$\text{Area} = \frac{1}{2}bh$$

$$= \frac{1}{2} \times \frac{12}{5} \times \frac{9}{5}$$

$$= \frac{54}{25} \text{ unit}^2$$

Question 4 (a)

$$\frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{\sin x \cdot 1 - x \cdot \cos x}{\sin^2 x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$

Question 4 (b)

$$\int_e^{e^3} \frac{5}{x}\,dx = \left[5\ln x\right]_e^{e^3}$$

$$= 5\ln e^3 - 5\ln e$$

$$= 5\times 3 - 5\times 1$$

$$= 10$$

Question 4 (c)

$\frac{dy}{dx} = 6x - 2$, through $(-1, 4)$

$$y = 3x^2 - 2x + C$$
$$4 = 3(-1)^2 - 2(-1) + C$$
$$4 = 5 + C$$
$$C = -1$$

$$\therefore\ y = 3x^2 - 2x - 1$$

Question 4 (d) (i)

$$y = \sqrt{9 - x^2} = \left(9 - x^2\right)^{\frac{1}{2}}$$

$$y' = \frac{1}{\cancel{2}}\left(9 - x^2\right)^{-\frac{1}{2}} \cdot \left(-\cancel{2}x\right)$$

$$= \frac{-x}{\sqrt{9 - x^2}}$$

Question 4 (d) (ii)

$$\int \frac{6x}{\sqrt{9 - x^2}}\,dx = -6\int \frac{-x}{\sqrt{9 - x^2}}\,dx$$

$$= -6\sqrt{9 - x^2} + C \qquad \text{(from part (i))}$$

Question 4 (e)

$y \le 4 - x^2$ and $y \ge |x| - 2$

Question 5 (a) (i)

$27, 54, \ldots$ GP $\quad (a = 27,\ r = 2)$

$$
\begin{aligned}
T_n &= ar^{n-1} \\
T_{12} &= 27 \times 2^{12-1} \\
&= 55\,296
\end{aligned}
$$

Question 5 (a) (ii)

$$
\begin{aligned}
n = ? \qquad T_n &> 10\,000\,000 \\
27 \times 2^{n-1} &> 10\,000\,000 \\
2^{n-1} &> \frac{10\,000\,000}{27} \\
n - 1 &> \frac{\ln\left(\dfrac{10\,000\,000}{27}\right)}{\ln 2} \\
n &> \frac{\ln\left(\dfrac{10\,000\,000}{27}\right)}{\ln 2} + 1 \\
n &> 19.49 \ldots \\
\therefore \quad n &= 20
\end{aligned}
$$

$\therefore$ On day 20.

Question 5 (a) (iii)

$$
\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \\
S_{12} &= \frac{27\left(2^{12} - 1\right)}{2 - 1}
\end{aligned}
$$

$\therefore$ Income $= 110\,565 \times \$0.005$

$= \$553.00$ (nearest dollar)

Question 5 (b) (i)

R — red shirt $\quad$ Y — yellow shirt

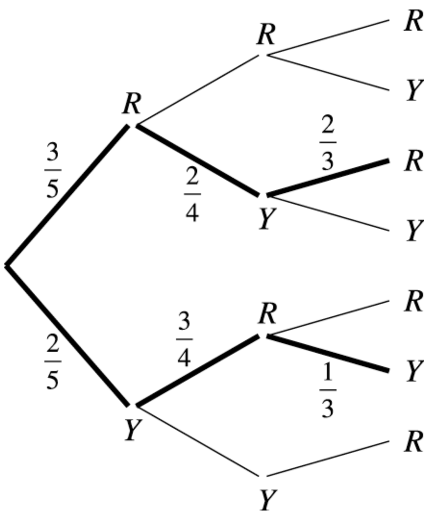
$$P(R) = \frac{3}{5}$$

Question 5 (b) (ii)

YYY is not possible.

So answer is $P(RRR) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$

Question 5 (b) (iii)



$$\begin{aligned} P(RYR \text{ or } YRY) &= \frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}+\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3} \\ &= \frac{18}{60} \\ &= \frac{3}{10} \end{aligned}$$

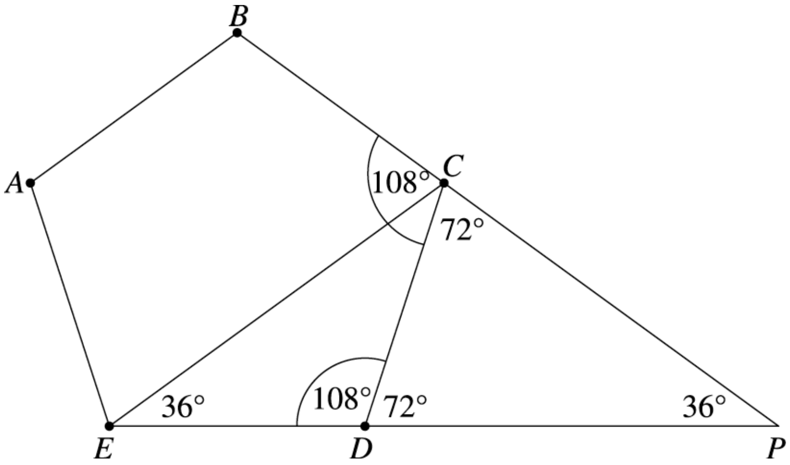
Question 5 (c)

$$\int_0^{20} v\,dt \doteqdot \frac{h}{3}\left[v_0 + 4v_1 + 2v_2 + 4v_3 + v_4\right]$$

$$= \frac{5}{3}\left[173 + 4\times 81 + 2\times 127 + 4\times 195 + 168\right]$$

$$\doteqdot 2831\frac{2}{3} \text{ metres}$$

Question 6 (a)



Question 6 (a) (i)

$$\angle CDE = \frac{(5-2)180°}{5} \quad \text{(angle in a regular polygon)}$$

$$= 3 \times 36°$$

$$= 108°$$

Question 6 (a) (ii)

$\triangle CDE$ is isosceles because $CD = ED$ (regular pentagon).

$$\text{So } \angle DEC = \angle DCE = \frac{180° - 108°}{2} = 36°$$

$$\angle CDP = 180° - 108° \text{ (straight angle)}$$

$$= 72°$$

$$\text{Also, } \angle BCD = 108° \text{ (angle in a regular pentagon)}$$

$$\therefore \angle DCP = 180° - 108° \text{ (straight angle)}$$

$$= 72°$$

$$\therefore \angle CPD = 180° - 2 \times 72° = 36°$$

$$= \angle DEC$$

This proves that $\triangle CEP$ has 2 equal angles $\therefore$ it is isosceles.

Question 6 (b)

$$PA^2 + PB^2 = 40$$

$$\left[(x+1)^2 + y^2\right] + \left[(x-3)^2 + y^2\right] = 40$$

$$x^2 + 2x + 1 + y^2 + x^2 - 6x + 9 + y^2 = 40$$

$$2x^2 - 4x + 2y^2 = 15$$

$$(x-1)^2 + y^2 = 16 \quad \text{(completion of square)}$$

$\therefore$ Circle $C(1,0)$, $r = 4$.

Question 6 (c) (i)

- $A(0,2)$

Question 6 (c) (ii)

$$\int_0^{\frac{\pi}{2}} 2\cos x\,dx = \left[2\sin x\right]_0^{\frac{\pi}{2}}$$

$$= 2\sin\frac{\pi}{2} - 2\sin 0$$

$$= 2 \times 1 - 0$$

$$= 2$$

Question 6 (c) (iii)

- C

Question 6 (c) (iv)

Area $= 4 \times 2$ (area A is 2 from (ii), areas A, C, $B/2$ are equal)
$= 8$ unit^2

Question 6 (c) (v)

$$\int_{\frac{\pi}{2}}^{2\pi} 2\cos x\,dx = 2 - 2 \times 2 \quad \text{(area } A \text{ minus area } B\text{)}$$

$$= -2$$

Question 7 (a) (i)

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

For stationary points $f'(x) = 0$

$3x^2 - 3 = 0$

$x^2 - 1 = 0$

$x = \pm 1$

at $x = 1$, $y = 0$
and $f''(1) = 6 > 0$

$\therefore$ $(1, 0)$ is a minimum turning point.

at $x = -1$, $y = 4$
and $f''(-1) = -6 < 0$

$\therefore$ $(-1, 4)$ is a maximum turning point.

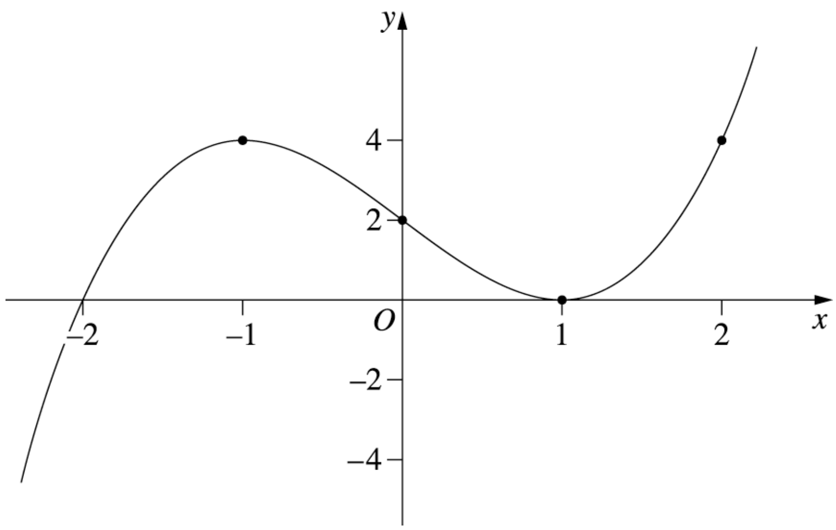
Question 7 (a) (ii)

$f(x) = x^3 - 3x + 2$

y-intercept $= 2$

at $x = 2$, $y = 4$

at $x = -2$, $y = 0$



Question 7 (b) (i)

$\dot{x} = 8 - 8e^{-2t}$

When $t = 0$,

$$\begin{aligned} \dot{x} &= 8 - 8e^{0} \\ &= 8 - 8 \\ &= 0 \end{aligned}$$

$\therefore$ particle is initially at rest.

Question 7 (b) (ii)

$\dot{x} = 8 - 8e^{-2t}$

$\ddot{x} = 16e^{-2t} > 0$ since $e^{-2t} > 0$ for all t

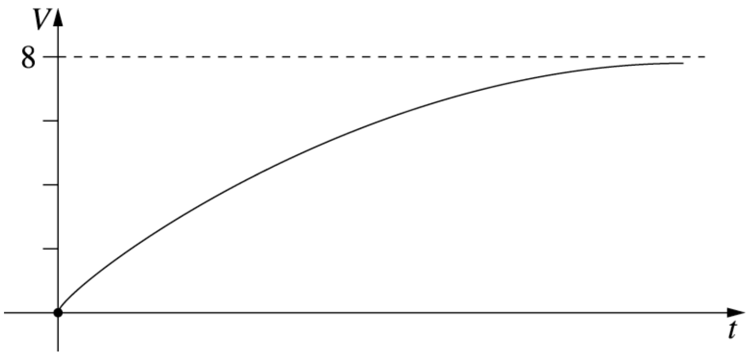
Question 7 (b) (iii)

Since the particle starts from rest and always has positive acceleration, its velocity will always be positive, ie travelling to the right. As $0 < e^{-2t} \leq 1$, we have $0 < 8e^{-2t} \leq 8$, and so $\dot{x} \geq 0$. Hence, the particle is always moving to the right.

Question 7 (b) (iv)

As $t \to \infty$, $\dot{x} \to 8$

Question 7 (b) (v)



Question 8 (a) (i)

By the cosine rule:

$$22^2 = x^2 + 20^2 - 2.20 \times \cos 60^\circ$$
$$484 = x^2 + 400 - 20x$$
$$0 = x^2 - 20x - 84$$

Question 8 (a) (ii)

Solve quadratic from part (i):

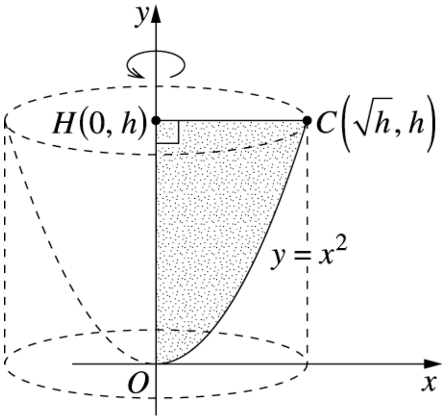
$$x = 10 \pm \sqrt{100 + 84}$$
$$= 10 \pm \sqrt{184}$$

Since the triangle is acute angled, ie $\angle LPS = 60^\circ$, only the positive solution applies.

$\therefore \quad x = 10 + \sqrt{184} \approx 24 \text{ km}$

is the distance.

Question 8 (b)



$y = x^2$

$x = \sqrt{y}$

Question 8 (b) (i)

$$V = \pi\int_0^h \left(\sqrt{y}\right)^2 dy \qquad \text{(volume of revolution)}$$

$$= \int_0^h y\,dy$$

$$= \pi\left[\frac{y^2}{2}\right]_0^h$$

$$= \frac{\pi h^2}{2}$$

Question 8 (b) (ii)

Volume of cylinder = $\pi r^2 h$

and C has coordinates $\left(\sqrt{h}, h\right)$
$\therefore$ radius $= \sqrt{h}$

$$\therefore \ \text{Vol} = \pi\left(\sqrt{h}\right)^2 h$$
$$= \pi h$$

$\therefore$ ratio of volume of the paraboloid to the volume of the cylinder

$$= \frac{\pi h^2}{2} : \pi h^2$$
$$= \frac{1}{2} : 1$$
$$= 1 : 2$$

Question 8 (c) (i)

Let A_n be amount in account at end of n months.

$$r = \frac{6\%}{12} = 0.005 \ , \ n = 420$$

$$A_1 = 100(1+0.005)^1 = 100(1.005)^1$$
$$A_2 = \left(100(1.005)+100\right)1.005$$
$$= 100(1.005)^2 + 100(1.005)^1$$
$$= 100\left(1.005^2 + 1.005\right)$$

$$A_{420} = 100\left(1.005 + 1.005^2 + \cdots + 1.005^{420}\right)$$
$$= 100 \times \frac{a\left(r^n - 1\right)}{r-1} \quad \text{where } r = 1.005,\ a = 1.005,\ n = 420$$
$$= 100 \times \frac{1.005\left(1.005^{420} - 1\right)}{1.005 - 1}$$
$$= \$143183 \text{ (nearest dollar)}$$

Question 8 (c) (ii) (1)

$$A_1 = (29\,227 + M)1.005$$

$$\begin{aligned} A_2 &= ((29\,227 + M)1.005 + M)1.005 \\ &= 29\,227 \times 1.005^2 + M\left(1.005^2 + 1.005\right) \qquad \text{as required} \end{aligned}$$

Question 8 (c) (ii) (2)

$$\begin{aligned} A_{240} &= 29\,227(1.005)^{240} + M\left(1.005 + 1.005^2 + \cdots + 1.005^{240}\right) \\ 800\,000 &= 29\,227(1.005)^{240} + M\left(\frac{1.005\left(1.005^{240} - 1\right)}{1.005 - 1}\right) \\ M &= \frac{\left[800\,000 - 29\,227(1.005)^{240}\right] \times 0.005}{1.005\left(1.005^{240} - 1\right)} \\ &\doteqdot \$1514.48 \end{aligned}$$

Question 9 (a) (i)

A line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

$$\therefore BC = \frac{1}{2}DE$$

The triangles are similar since the corresponding sides are in proportion.

Question 9 (a) (ii)

$\triangle BFC$ is similar to $\triangle EFD$ since $BC \parallel DE$ (alternate angles, so the triangles are equiangular).

$\therefore$ the corresponding sides are in ratio.

From part (i) $BC : DE = 1:2$, so $BF : FE = 1:2$ as well.

Question 9 (b) (i)

The difference in the rates is:

$$\left(2+\frac{t^2}{t+1}\right)-\left(1+\frac{1}{t+1}\right)$$

$$= 1+\frac{t^2}{t+1}-\frac{1}{t+1}$$

$$= 1+\frac{t^2-1}{t+1}$$

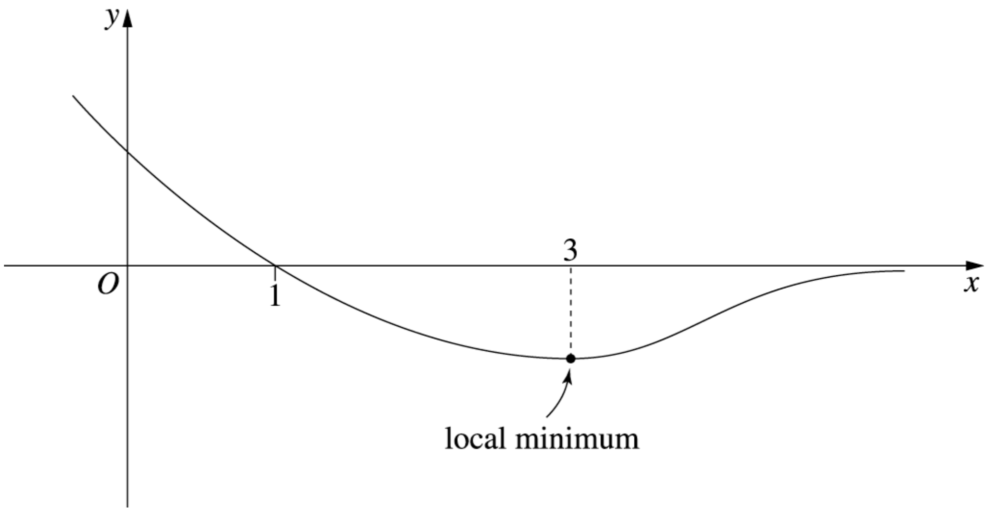
$$= 1+\frac{(t+1)(t-1)}{t+1} = 1+t-1 = t$$

Question 9 (b) (ii)

The difference in volume is:

$$\int_0^4 t\,dt = \left.\frac{t^2}{2}\right|_0^4 = \frac{16}{2} = 18 \text{ litres}$$

Question 9 (c)



Question 9 (d) (i)

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}-\sqrt{n}\right)}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{n+1-n} = \sqrt{n+1}-\sqrt{n}$$

Question 9 (d) (ii)

Using part (i):

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{98}+\sqrt{99}}+\frac{1}{\sqrt{99}+\sqrt{100}}$$

$$= \left(\cancel{\sqrt{2}}-\sqrt{1}\right)+\left(\cancel{\sqrt{3}}-\cancel{\sqrt{2}}\right)+\left(\cancel{\sqrt{4}}-\cancel{\sqrt{3}}\right)+\cdots+\left(\cancel{\sqrt{99}}+\cancel{\sqrt{98}}\right)+\left(\sqrt{100}+\cancel{\sqrt{99}}\right)$$

$$= -\sqrt{1}+\sqrt{100} = -1+10$$

$$= 9$$

Question 10 (a) (i)

$$I = 10^{-12}e^{\frac{110}{10}} = 10^{-12}e^{11}$$
$$\approx 5.99\times 10^{-8}$$

Question 10 (a) (ii)

$$8.1\times 10^{-9} = 10^{-12}e^{\frac{110}{10}0.1L}$$
$$8.1\times 10^{3} = e^{\frac{L}{10}}$$
$$\ln 8100 = \frac{L}{10}$$
$$L = 10\ln 8100 \approx 89.996$$
$$= 90 \text{ decibels}$$

Question 10 (a) (iii)

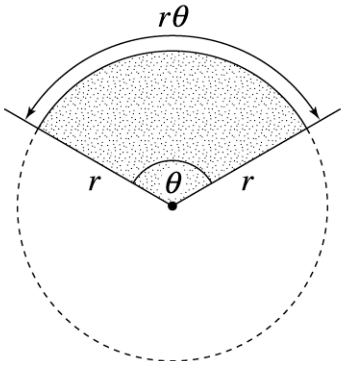
Consider two intensities:

$$I_1 = 10^{-12}e^{\frac{L_1}{10}}, \qquad I_2 = 10^{-12}e^{\frac{L_2}{10}}$$

and assume $I_2 = 2I_1$. Hence,

$$I_2 = 10^{-12}e^{\frac{L_2}{10}} \qquad = 2\times 10^{-12}e^{\frac{L_1}{10}} = 2I_1$$
$$e^{\frac{L_2}{10}} = 2e^{\frac{L_1}{10}}$$
$$\frac{L_2}{10} = \ln\left(2e^{\frac{L_1}{10}}\right) = \ln 2 + \frac{L_1}{10}$$
$$L_2 - L_1 = 10\ln 2$$

Question 10 (b) (i)



Length of arc $r\theta$

Two radial pieces of length r

Total length $P = r\theta + 2r = r(\theta + 2)$

Question 10 (b) (ii)

From part (i)

$$\theta = \frac{1}{r}(P - 2r)$$

The area of the paddock (sector) is:

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r(P - 2r) = \frac{1}{2}Pr - r^2$$

Question 10 (b) (iii)

From part (ii)

$$\frac{dA}{dr} = \frac{P}{2} - 2r = 2\left(\frac{P}{4} - r\right)$$

$$\therefore \ \frac{dA}{dr} = 0 \text{ if } \frac{P}{4} = r$$

A is a maximus for $r = \dfrac{P}{4}$ since

$$\frac{dA}{dr} = 2\left(\frac{P}{4} - r\right) > 0 \text{ if } r < \frac{P}{4},$$

$$\frac{dA}{dr} = 2\left(\frac{P}{4} - r\right) < 0 \text{ if } r < \frac{P}{2},$$

so A increases if $r < \dfrac{P}{4}$ and decreases if $r > \dfrac{P}{4}$.

Question 10 (b) (iv)

Substitute $r = \frac{P}{4}$ into $P = r(\theta + 2)$

$P = \frac{P}{4}(\theta + 2)$

$4 = \theta + 2$

$\theta = 2$

Question 10 (b) (v)

From part (i)

$P = r(\theta + 2) > 2r$, so

$r < \frac{P}{2}$

From part (i) since $\theta < 2\pi$

$P = r(\theta + 2) < r(2\pi + 2)$, so

$r > \frac{P}{2(\pi + 1)}$

$\therefore \frac{P}{2(\pi + 1)} < r < \frac{p}{2}$