NSW Education Standards Authority

2018 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–16)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the value of $7^{-1.3}$ correct to two decimal places?

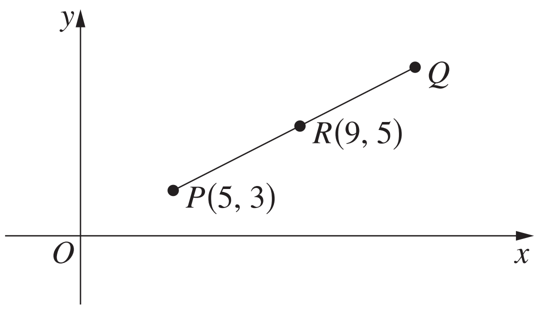
A. 0.07

B. 0.08

C. −12.54

D. −12.55

2 The point $R(9, 5)$ is the midpoint of the interval PQ, where P has coordinates $(5, 3)$.



NOT TO SCALE

What are the coordinates of Q?

A. (4, 7)

B. (7, 4)

C. (13, 7)

D. (14, 8)

3 What is the x-intercept of the line $x + 3y + 6 = 0$?

A. (−6, 0)

B. (6, 0)

C. (0, −2)

D. (0, 2)

4 The line $3x - 4y + 3 = 0$ is a tangent to a circle with centre $(3, -2)$.

What is the equation of the circle?

A. $(x+3)^2 + (y-2)^2 = 4$

B. $(x-3)^2 + (y+2)^2 = 4$

C. $(x+3)^2 + (y-2)^2 = 16$

D. $(x-3)^2 + (y+2)^2 = 16$

5 What is the derivative of $\sin(\ln x)$, where $x > 0$?

A. $\cos\left(\frac{1}{x}\right)$

B. $\cos(\ln x)$

C. $\cos\left(\frac{\ln x}{x}\right)$

D. $\frac{\cos(\ln x)}{x}$

6 A runner has four different pairs of shoes.

If two shoes are selected at random, what is the probability that they will be a matching pair?

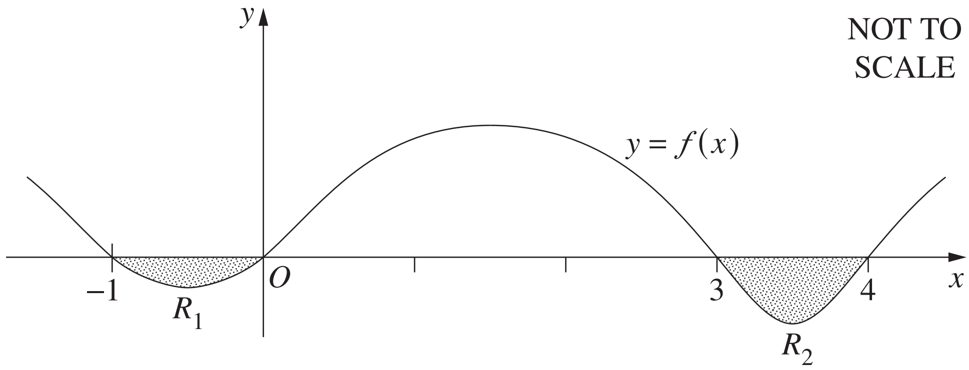
A. $\frac{1}{56}$

B. $\frac{1}{16}$

C. $\frac{1}{7}$

D. $\frac{1}{4}$

7 The diagram shows the graph of $y = f(x)$ with intercepts at $x = -1$, 0, 3 and 4.



The area of shaded region R_1 is 2.

The area of shaded region R_2 is 3.

It is given that $\int_0^4 f(x)\,dx = 10$.

What is the value of $\int_{-1}^{3} f(x)\,dx$?

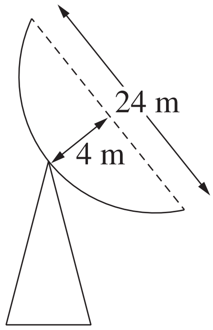
A. 5

B. 9

C. 11

D. 15

8 A radio telescope has a parabolic dish. The width of the opening is 24 m and the distance along the axis from the vertex to the opening is 4 m, as shown in the diagram.



What is the focal length of the parabola?

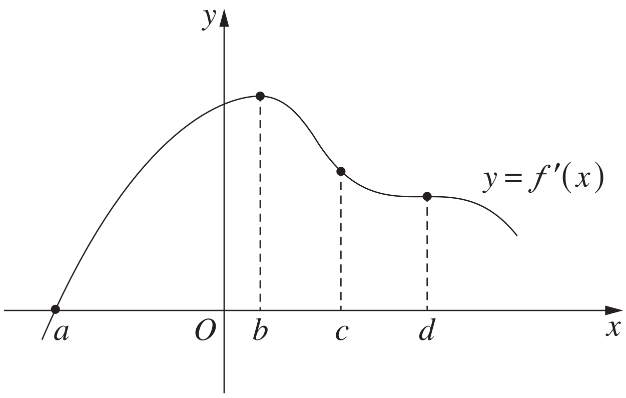
A. $\frac{1}{6}$ m

B. $\frac{1}{3}$ m

C. 6 m

D. 9 m

9 The diagram shows the graph of $f'(x)$, the derivative of a function.



For what value of x does the graph of the function $f(x)$ have a point of inflexion?

A. $x = a$

B. $x = b$

C. $x = c$

D. $x = d$

10 A trigonometric function $f(x)$ satisfies the condition

$$\int_0^{\pi} f(x)\,dx \neq \int_{\pi}^{2\pi} f(x)\,dx\,.$$

Which function could be $f(x)$?

A. $f(x) = \sin(2x)$

B. $f(x) = \cos(2x)$

C. $f(x) = \sin\left(\frac{x}{2}\right)$

D. $f(x) = \cos\left(\frac{x}{2}\right)$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Rationalise the denominator of $\dfrac{3}{3+\sqrt{2}}$. **2**

(b) Solve $1 - 3x > 10$. **2**

(c) Simplify $\dfrac{8x^3 - 27y^3}{2x - 3y}$. **2**

(d) In an arithmetic series, the third term is 8 and the twentieth term is 59.

(i) Find the common difference. **1**

(ii) Find the 50th term. **2**

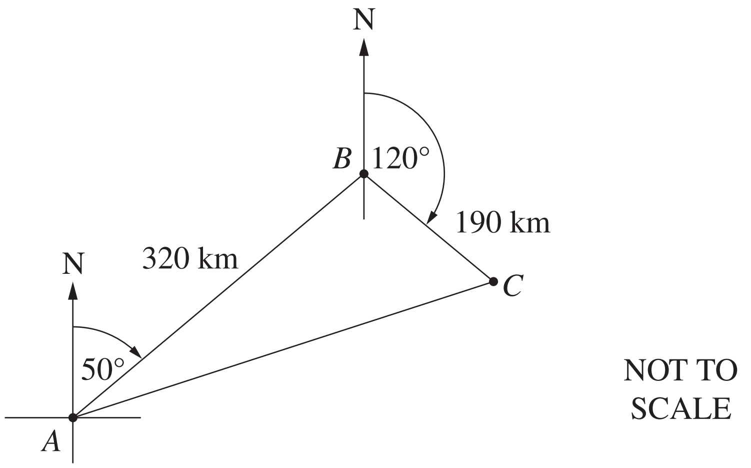
(e) Evaluate $\displaystyle\int_0^3 e^{5x}\,dx$. **2**

(f) Differentiate $x^2 \tan x$. **2**

(g) Differentiate $\dfrac{e^x}{x+1}$. **2**

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) A ship travels from Port A on a bearing of 050° for 320 km to Port B. It then travels on a bearing of 120° for 190 km to Port C.



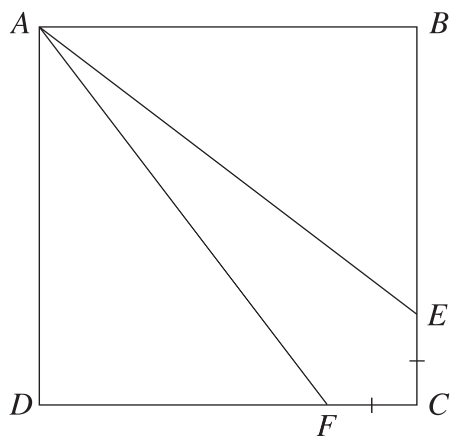
(i) What is the size of $\angle ABC$? **1**

(ii) What is the distance from Port A to Port C? Answer to the nearest 10 kilometres. **2**

(b) Find the equation of the tangent to the curve $y = \cos 2x$ at $x = \frac{\pi}{6}$. **3**

Question 12 continues on page 9

(c) The diagram shows the square $ABCD$. The point E is chosen on BC and the point F is chosen on CD so that $EC = FC$.



(i) Prove that $\triangle ADF$ is congruent to $\triangle ABE$. **2**

(ii) The side length of the square is 14 cm and EC has length 4 cm. Find the area of $AECF$. **2**

(d) The displacement of a particle moving along the x-axis is given by

$$x = \frac{t^3}{3} - 2t^2 + 3t,$$

where x is the displacement from the origin in metres and t is the time in seconds, for $t \geq 0$.

(i) What is the initial velocity of the particle? **1**

(ii) At which times is the particle stationary? **2**

(iii) Find the position of the particle when the acceleration is zero. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

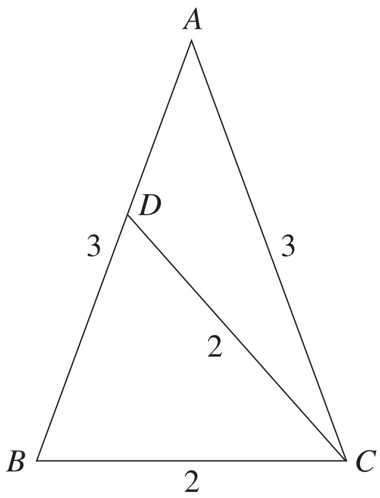
(a) Consider the curve $y = 6x^2 - x^3$.

(i) Find the stationary points and determine their nature. **3**

(ii) Given that the point (2, 16) lies on the curve, show that it is a point of inflexion. **2**

(iii) Sketch the curve, showing the stationary points, the point of inflexion and the x and y intercepts. **2**

(b) In $\triangle ABC$, sides AB and AC have length 3, and BC has length 2. The point D is chosen on AB so that DC has length 2.



(i) Prove that $\triangle ABC$ and $\triangle CBD$ are similar. **2**

(ii) Find the length AD. **2**

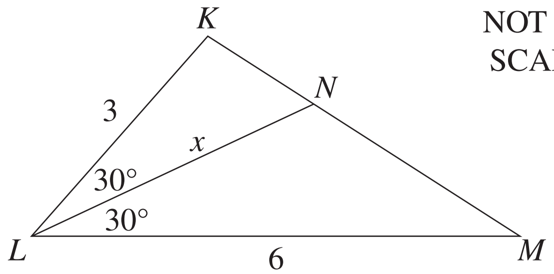
(c) The population of a country grew exponentially between 1910 and 2010. This population can be modelled by the equation $P(t) = 92e^{kt}$, where $P(t)$ is the population of the country in millions, t is the time in years after 1910 and k is a positive constant. The population of the country in 1960 was 184 million.

(i) Show that the value of k is 0.0139, correct to 4 decimal places. **2**

(ii) Assuming that this model continues to be valid after 2010, estimate the population of the country in 2020 to the nearest million. **2**

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) In $\triangle KLM$, KL has length 3, LM has length 6 and $\angle KLM$ is 60°. The point N is chosen on side KM so that LN bisects $\angle KLM$. The length LN is x.

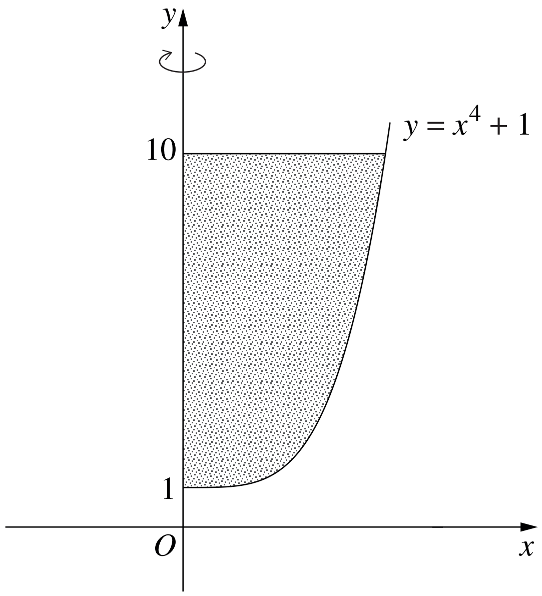


NOT TO SCALE

(i) Find the exact value of the area of $\triangle KLM$. **1**

(ii) Hence, or otherwise, find the exact value of x. **2**

(b) The shaded region shown in the diagram is bounded by the curve $y = x^4 + 1$, the y-axis and the line $y = 10$. **3**



NOT TO SCALE

Find the volume of the solid of revolution formed when the shaded region is rotated about the y-axis.

Question 14 continues on page 12

Question 14 (continued)

(c) Let $f(x) = x^3 + kx^2 + 3x - 5$, where k is a constant. **3**

Find the values of k for which $f(x)$ has NO stationary points.

(d) An artist posted a song online. Each day there were $2^n + n$ downloads, where n is the number of days after the song was posted.

(i) Find the number of downloads on each of the first 3 days after the song was posted. **1**

(ii) What is the total number of times the song was downloaded in the first 20 days after it was posted? **2**

(e) Two machines, A and B, produce pens. It is known that 10% of the pens produced by machine A are faulty and that 5% of the pens produced by machine B are faulty.

(i) One pen is chosen at random from each machine. **1**

What is the probability that at least one of the pens is faulty?

(ii) A coin is tossed to select one of the two machines. Two pens are chosen at random from the selected machine. **2**

What is the probability that neither pen is faulty?

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) The length of daylight, $L(t)$, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation

$$L(t) = 12 + 2\cos\left(\frac{2\pi t}{366}\right),$$

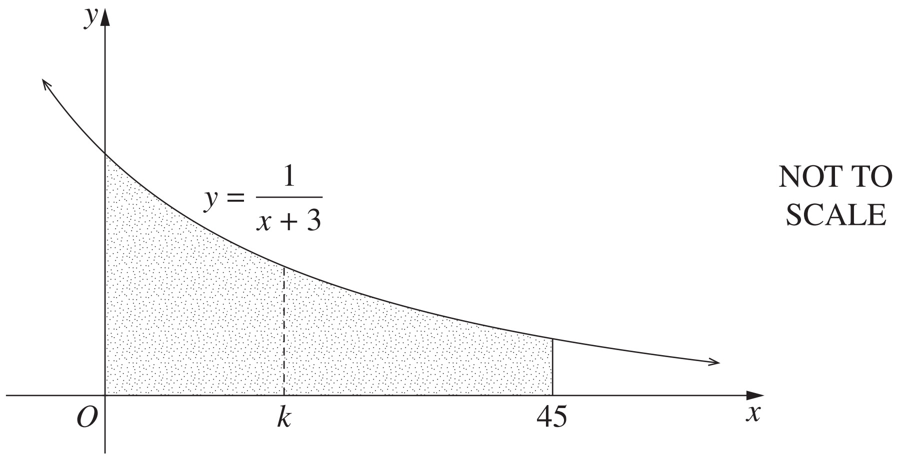
where t is the number of days after 21 December 2015, for $0 \le t \le 366$.

(i) Find the length of daylight on 21 December 2015. **1**

(ii) What is the shortest length of daylight? **1**

(iii) What are the two values of t for which the length of daylight is 11? **2**

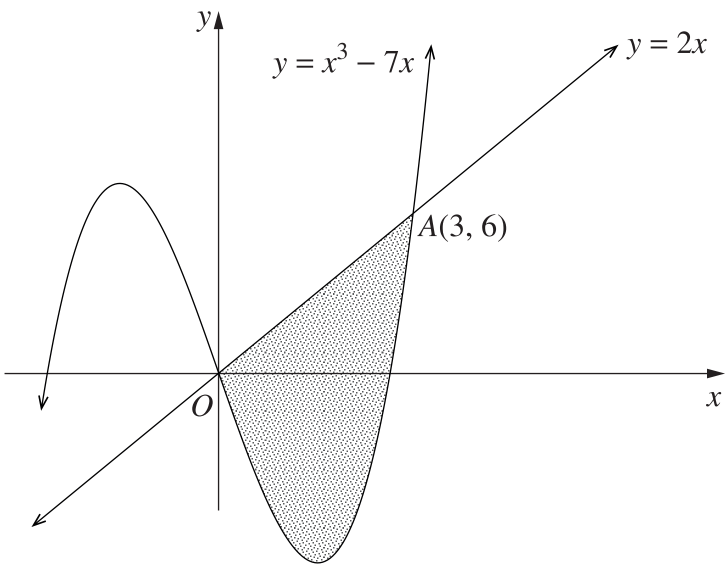
(b) The diagram shows the region bounded by the curve $y = \dfrac{1}{x+3}$ and the lines $x = 0$, $x = 45$ and $y = 0$. The region is divided into two parts of equal area by the line $x = k$, where k is a positive integer. **3**



What is the value of the integer k, given that the two parts have equal areas?

Question 15 continues on page 14

(c) The shaded region is enclosed by the curve $y = x^3 - 7x$ and the line $y = 2x$, as shown in the diagram. The line $y = 2x$ meets the curve $y = x^3 - 7x$ at $O(0, 0)$ and $A(3, 6)$. Do NOT prove this.



NOT TO SCALE

(i) Use integration to find the area of the shaded region. **2**

(ii) Verify that one application of Simpson's rule gives the exact area of the shaded region. **2**

The point P is chosen on the curve $y = x^3 - 7x$ so that the tangent at P is parallel to the line $y = 2x$ and the x-coordinate of P is positive.

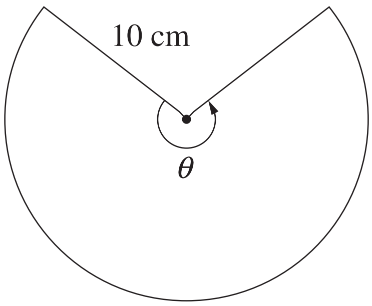
(iii) Show that the coordinates of P are $\left(\sqrt{3}, -4\sqrt{3}\right)$. **2**

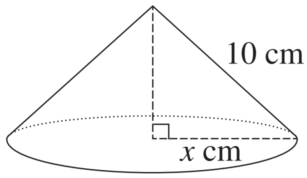
(iv) Find the area of $\triangle OAP$. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) A sector with radius 10 cm and angle θ is used to form the curved surface of a cone with base radius x cm, as shown in the diagram.





NOT TO SCALE

The volume of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.

(i) Show that the volume, V cm^3, of the cone described above is given by **1**

$$V = \frac{1}{3}\pi x^2 \sqrt{100 - x^2}\,.$$

(ii) Show that $\dfrac{dV}{dx} = \dfrac{\pi x\left(200 - 3x^2\right)}{3\sqrt{100 - x^2}}$. **2**

(iii) Find the exact value of θ for which V is a maximum. **3**

(b) A game involves rolling two six-sided dice, followed by rolling a third six-sided die. To win the game, the number rolled on the third die must lie between the two numbers rolled previously. For example, if the first two dice show 1 and 4, the game can only be won by rolling a 2 or 3 with the third die.

(i) What is the probability that a player has no chance of winning before rolling the third die? **2**

(ii) What is the probability that a player wins the game? **2**

Question 16 continues on page 16

(c) Kara deposits an amount of \$300 000 into an account which pays compound interest of 4% per annum, added to the account at the end of each year. Immediately after the interest is added, Kara makes a withdrawal for expenses for the coming year. The first withdrawal is \$*P*. Each subsequent withdrawal is 5% greater than the previous one.

Let \$$A_n$ be the amount in the account after the *n*th withdrawal.

(i) Show that $A_2 = 300\,000(1.04)^2 - P\big[(1.04) + (1.05)\big]$. **1**

(ii) Show that $A_3 = 300\,000(1.04)^3 - P\Big[(1.04)^2 + (1.04)(1.05) + (1.05)^2\Big]$. **1**

(iii) Show that there will be money in the account when **3**

$$\left(\frac{105}{104}\right)^n < 1 + \frac{3000}{P}.$$

End of paper

NSW Education Standards Authority

2018 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

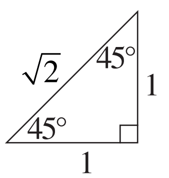
$$\sec\theta = \frac{1}{\cos\theta}$$

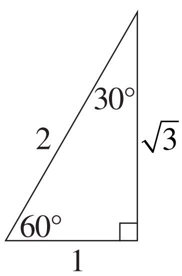
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios





Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point–gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\dfrac{dy}{dx} = F'(u)\dfrac{du}{dx}$

If $y = e^{f(x)}$, then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$\text{Area} = \dfrac{1}{2}r^2\theta$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \ \frac{my_2 + ny_1}{m+n} \right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at, at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1, y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0, y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos(nt + \alpha)$$

$$\ddot{x} = -n^2(x - b)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$