BOARD OF STUDIES
NEW SOUTH WALES

2006

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a)   Evaluate $e^{-0.5}$ correct to three decimal places. **2**

(b)   Factorise $2x^2 + 5x - 3$. **2**

(c)   Sketch the graph of $y = |x + 4|$. **2**

(d)
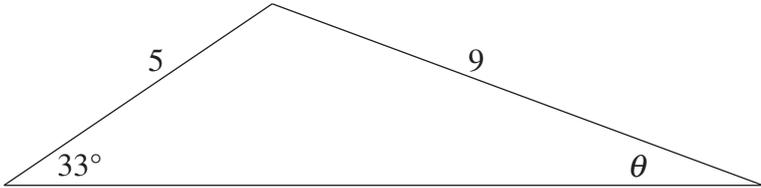
NOT TO SCALE **2**

Find the value of θ in the diagram. Give your answer to the nearest degree.

(e)   Solve $3 - 5x \leq 2$. **2**

(f)   Find the limiting sum of the geometric series $\dfrac{13}{5} + \dfrac{13}{25} + \dfrac{13}{125} + \cdots$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

    (i) $x \tan x$ **2**
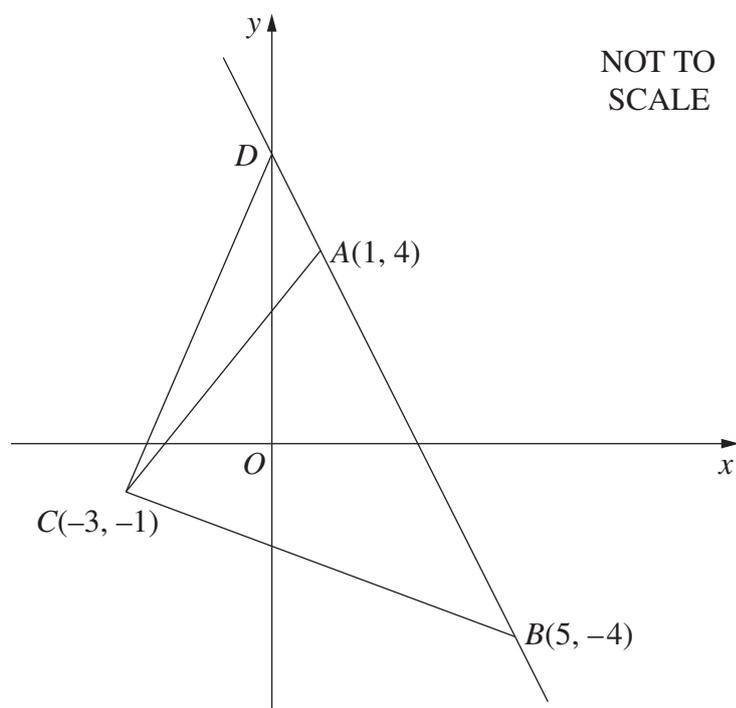
    (ii) $\dfrac{\sin x}{x+1}$.   **2**

(b)   (i) Find $\displaystyle\int 1 + e^{7x}\, dx$.   **2**

    (ii) Evaluate $\displaystyle\int_0^3 \dfrac{8x}{1+x^2}\, dx$.   **3**

(c) Find the equation of the tangent to the curve $y = \cos 2x$ at the point whose x-coordinate is $\dfrac{\pi}{6}$.   **3**

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a)

In the diagram, A, B and C are the points $(1, 4)$, $(5, -4)$ and $(-3, -1)$ respectively. The line AB meets the y-axis at D.

(i) Show that the equation of the line AB is $2x + y - 6 = 0$. **2**

(ii) Find the coordinates of the point D. **1**

(iii) Find the perpendicular distance of the point C from the line AB. **1**

(iv) Hence, or otherwise, find the area of the triangle ADC. **2**

Question 3 (continued)

(b)   Evaluate $\displaystyle\sum_{r=2}^{4} \frac{1}{r}$.

1

(c)   On the first day of the harvest, an orchard produces 560 kg of fruit. On the next day, the orchard produces 543 kg, and the amount produced continues to decrease by the same amount each day.

(i)   How much fruit is produced on the fourteenth day of the harvest?

2

(ii)   What is the total amount of fruit that is produced in the first 14 days of the harvest?

1

(iii)   On what day does the daily production first fall below 60 kg?

2

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a)
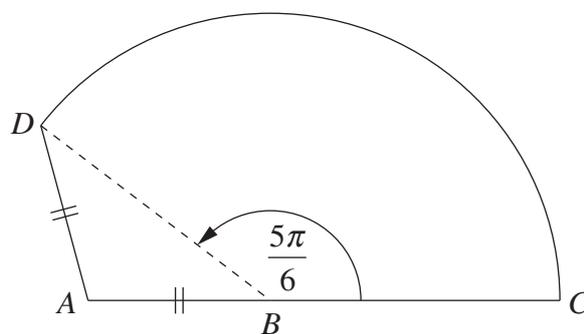
NOT TO
SCALE

In the diagram, *ABCD* represents a garden. The sector *BCD* has centre *B* and $\angle DBC = \dfrac{5\pi}{6}$.

The points *A*, *B* and *C* lie on a straight line and $AB = AD = 3$ metres.

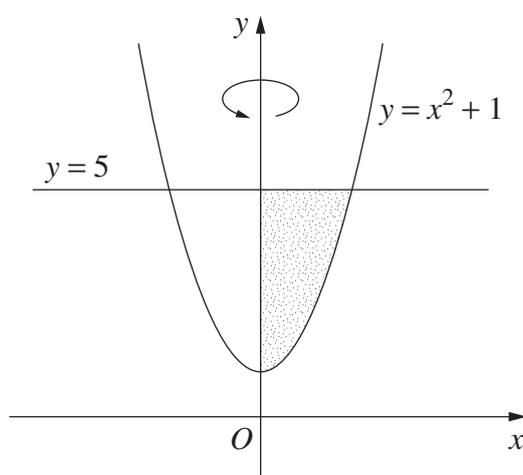Copy or trace the diagram into your writing booklet.

  (i)   Show that $\angle DAB = \dfrac{2\pi}{3}$.   **1**

  (ii)   Find the length of *BD*.   **2**

  (iii)   Find the area of the garden *ABCD*.   **2**

Question 4 continues on page 7

Question 4 (continued)

(b) **3**

In the diagram, the shaded region is bounded by the parabola $y = x^2 + 1$, the y-axis and the line $y = 5$.

Find the volume of the solid formed when the shaded region is rotated about the y-axis.

(c) A chessboard has 32 black squares and 32 white squares. Tanya chooses three different squares at random.

   (i)   What is the probability that Tanya chooses three white squares? **2**

   (ii) What is the probability that the three squares Tanya chooses are the same colour? **1**

   (iii) What is the probability that the three squares Tanya chooses are not the same colour? **1**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)   A function $f(x)$ is defined by $f(x) = 2x^2(3-x)$.

    (i)   Find the coordinates of the turning points of $y = f(x)$ and determine their nature.   **3**

    (ii)   Find the coordinates of the point of inflexion.   **1**

    (iii)   Hence sketch the graph of $y = f(x)$, showing the turning points, the point of inflexion and the points where the curve meets the x-axis.   **3**

    (iv)   What is the minimum value of $f(x)$ for $-1 \le x \le 4$?   **1**

(b)   (i)   Show that $\dfrac{d}{dx}\log_e(\cos x) = -\tan x$.   **1**
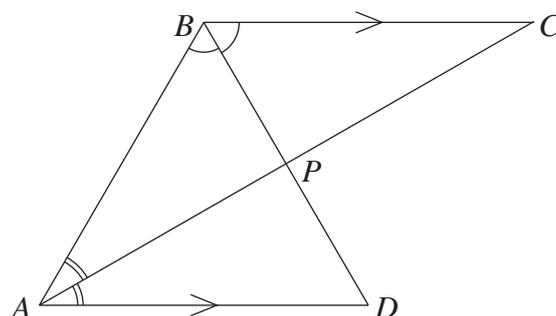
    (ii)   **3**
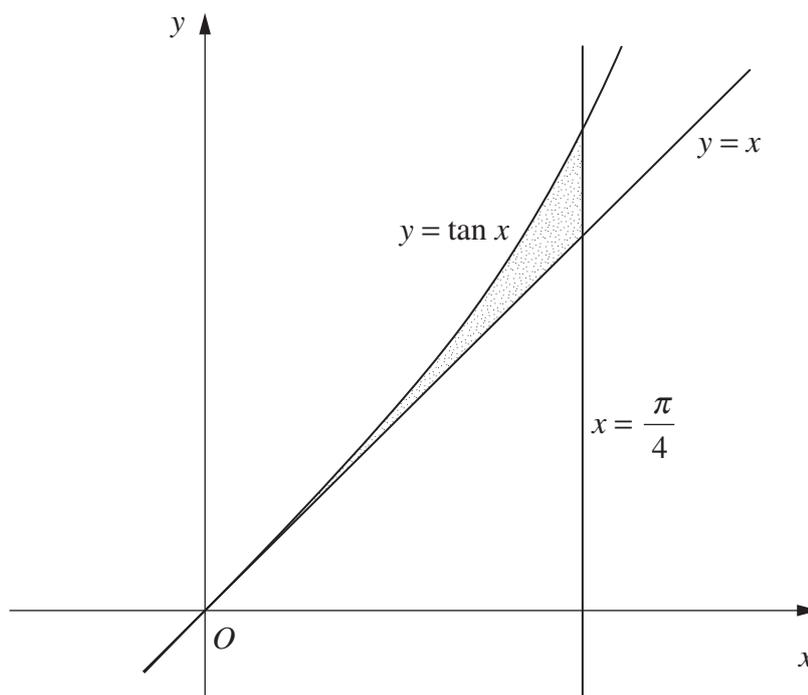
The shaded region in the diagram is bounded by the curve $y = \tan x$ and the lines $y = x$ and $x = \dfrac{\pi}{4}$.

Using the result of part (i), or otherwise, find the area of the shaded region.

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a)

In the diagram, *AD* is parallel to *BC*, *AC* bisects ∠*BAD* and *BD* bisects ∠*ABC*. The lines *AC* and *BD* intersect at *P*.

Copy or trace the diagram into your writing booklet.

   (i)   Prove that ∠*BAC* = ∠*BCA*.　　　　　　　　　　　　　　　　　　**1**

   (ii)   Prove that △*ABP* ≡ △*CBP*.　　　　　　　　　　　　　　　　**2**

   (iii)   Prove that *ABCD* is a rhombus.　　　　　　　　　　　　　　　　**3**

(b)   A rare species of bird lives only on a remote island. A mathematical model predicts that the bird population, *P*, is given by

$$P = 150 + 300e^{-0.05t}$$

where *t* is the number of years after observations began.

   (i)   According to the model, how many birds were there when observations began?　　　　　　**1**

   (ii)   According to the model, what will be the rate of change in the bird population ten years after observations began?　　　　　　**2**

   (iii)   What does the model predict will be the limiting value of the bird population?　　　　　　**1**

   (iv)   The species will become eligible for inclusion in the endangered species list when the population falls below 200. When does the model predict that this will occur?　　　　　　**2**

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Let α and β be the solutions of $x^2 - 3x + 1 = 0$.

    (i) Find $\alpha\beta$. **1**

    (ii) Hence find $\alpha + \dfrac{1}{\alpha}$. **1**

(b) A function $f(x)$ is defined by $f(x) = 1 + 2\cos x$.

    (i) Show that the graph of $y = f(x)$ cuts the x-axis at $x = \dfrac{2\pi}{3}$. **1**

    (ii) Sketch the graph of $y = f(x)$ for $-\pi \le x \le \pi$ showing where the graph cuts each of the axes. **3**

    (iii) Find the area under the curve $y = f(x)$ between $x = -\dfrac{\pi}{2}$ and $x = \dfrac{2\pi}{3}$. **3**

(c) (i) Write down the discriminant of $2x^2 + (k - 2)x + 8$, where k is a constant. **1**

    (ii) Hence, or otherwise, find the values of k for which the parabola $y = 2x^2 + kx + 9$ does not intersect the line $y = 2x + 1$. **2**

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) A particle is moving in a straight line. Its displacement, x metres, from the origin, O, at time t seconds, where $t \geq 0$, is given by $x = 1 - \dfrac{7}{t+4}$.

    (i) Find the initial displacement of the particle. **1**

    (ii) Find the velocity of the particle as it passes through the origin. **3**

    (iii) Show that the acceleration of the particle is always negative. **1**

    (iv) Sketch the graph of the displacement of the particle as a function of time. **2**

(b) Joe borrows $200 000 which is to be repaid in equal monthly instalments. The interest rate is 7.2% per annum reducible, calculated monthly.

It can be shown that the amount, A_n, owing after the nth repayment is given by the formula:

$$A_n = 200\,000\,r^n - M\big(1 + r + r^2 + \cdots + r^{n-1}\big),$$

where $r = 1.006$ and M is the monthly repayment.   (Do NOT show this.)

    (i) The minimum monthly repayment is the amount required to repay the loan in 300 instalments. **3**

Find the minimum monthly repayment.

    (ii) Joe decides to make repayments of $2800 each month from the start of the loan. **2**

How many months will it take for Joe to repay the loan?

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Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Find the coordinates of the focus of the parabola $12y = x^2 - 6x - 3$. **2**

(b) During a storm, water flows into a 7000-litre tank at a rate of $\dfrac{dV}{dt}$ litres

per minute, where $\dfrac{dV}{dt} = 120 + 26t - t^2$ and t is the time in minutes
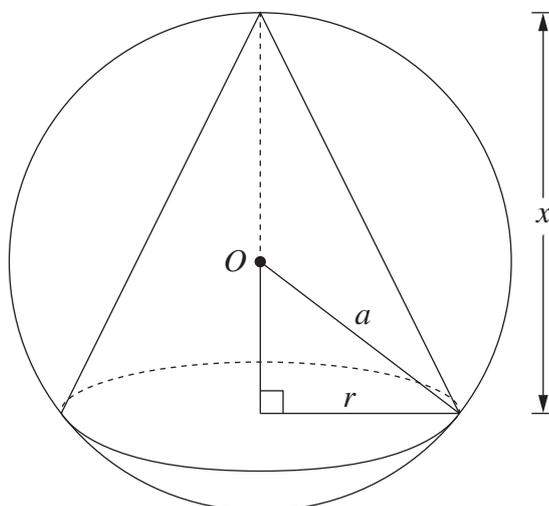
since the storm began.

    (i) At what times is the tank filling at twice the initial rate? **2**

    (ii) Find the volume of water that has flowed into the tank since the start of the storm as a function of t. **1**

    (iii) Initially, the tank contains 1500 litres of water. When the storm finishes, 30 minutes after it began, the tank is overflowing. **2**

    How many litres of water have been lost?

(c)

A cone is inscribed in a sphere of radius a, centred at O. The height of the cone is x and the radius of the base is r, as shown in the diagram.

    (i) Show that the volume, V, of the cone is given by $V = \dfrac{1}{3}\pi\left(2ax^2 - x^3\right)$. **2**

    (ii) Find the value of x for which the volume of the cone is a maximum. You must give reasons why your value of x gives the maximum volume. **3**

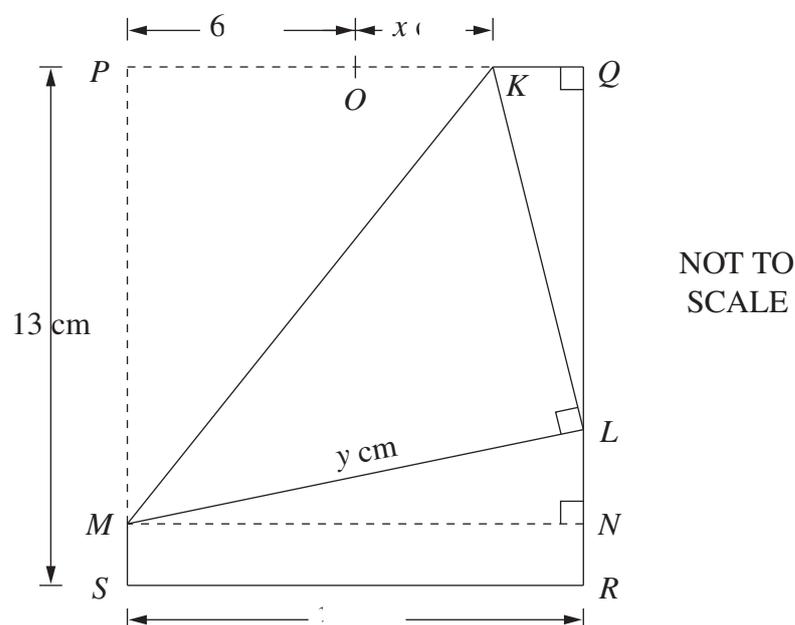**Question 10** (12 marks) Use a SEPARATE writing booklet.

(a) Use Simpson's rule with three function values to find an approximation to the **2** value of $\displaystyle\int_{0.5}^{1.5} \left(\log_e x\right)^3 dx$.

Give your answer correct to three decimal places.

(b)

A rectangular piece of paper *PQRS* has sides *PQ* = 12 cm and *PS* = 13 cm. The point *O* is the midpoint of *PQ*. The points *K* and *M* are to be chosen on *OQ* and *PS* respectively, so that when the paper is folded along *KM*, the corner that was at *P* lands on the edge *QR* at *L*. Let *OK* = *x* cm and *LM* = *y* cm.

Copy or trace the diagram into your writing booklet.

(i)   Show that $QL^2 = 24x$. **1**

(ii) Let *N* be the point on *QR* for which *MN* is perpendicular to *QR*. **3**
By showing that $\triangle QKL \;|||\; \triangle NLM$, deduce that $y = \dfrac{\sqrt{6}\,(6+x)}{\sqrt{x}}$.

(iii) Show that the area, *A*, of $\triangle KLM$ is given by $A = \dfrac{\sqrt{6}\,(6+x)^2}{2\sqrt{x}}$. **1**

(iv) Use the fact that $12 \le y \le 13$ to find the possible values of *x*. **2**

(v)   Find the minimum possible area of $\triangle KLM$. **3**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx \qquad = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx \qquad = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx \qquad = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx \qquad = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx \qquad = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx \qquad = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx \quad = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx \qquad = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx \qquad = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx \qquad = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx \qquad = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$